Boundary K-matrices for the six vertex and the $n(2n-1)A_{n-1}$ vertex models

1993 J. Phys. A: Math. Gen. 26 L519

(http://iopscience.iop.org/0305-4470/26/12/007)

J. Phys. A: Math. Gen. 26 (1993) L519–L524. Printed in the UK

LETTER TO THE EDITOR

Boundary K-matrices for the six vertex and the $n(2n-1)$ A_{n-1} vertex models

H J de Vega†§ and A González Ruiz‡

† LPTHE, Tour 16, ler. étage, Université Paris VI, 4, Place Jussieu, 75 252 Paris cedex 05, France and Isaac Newton Institute, 20 Clarkson Road, Cambridge, CB3 0EH, UK
‡ Departamento de Física Teórica, Universidad Complutense, 28040 Madrid, Spain

Received 27 November 1992

Abstract. Boundary conditions compatible with integrability are obtained for two-dimensional models by solving the factorizability equations for the reflection matrices $K^{\pm}(\theta)$. For the six vertex model the general solution depending on four arbitrary parameters is found. For the A_{n-1} models all diagonal solutions are found. The associated integrable magnetic Hamiltonians are explicitly derived.

As is by now well known integrability is a consequence of the Yang–Baxter equation in two-dimensional lattice models and quantum field theory [1]. The Yang–Baxter equation takes the form:

$$[1\otimes R(\theta-\theta')][R(\theta)\otimes 1][1\otimes R(\theta')]=[R(\theta')\otimes 1][1\otimes R(\theta)][R(\theta-\theta')\otimes 1] \tag{1}$$

where the R matrix elements $R^{ab}_{cd}(\theta)$ with $(1\leqslant a,b,c,d\leqslant n, n\geqslant 2)$ define the statistical weights for a vertex model in two dimensions.

Not all boundary conditions (BC) are compatible with integrability in the bulk. Equation (1) guarantees the integrability on the bulk. Integrability holds for BC defined by matrices $K^{-}(\theta)$ and $K^{+}(\theta)$ (associated to the left and right boundaries respectively) provided the R matrix has P, T and crossing symmetry and the following equations proposed by Cherednik and Sklyanin [2] are fulfilled:

$$R(\theta-\theta')[K^{-}(\theta)\otimes 1]R(\theta+\theta')[K^{-}(\theta')\otimes 1]=[K^{-}(\theta')\otimes 1]R(\theta+\theta')[K^{-}(\theta)\otimes 1]R(\theta-\theta') \tag{2}$$

$$R(\theta-\theta')[1\otimes K^{+}(\theta)]R(\theta+\theta')[1\otimes K^{+}(\theta')]=[1\otimes K^{+}(\theta')]R(\theta+\theta')[1\otimes K^{+}(\theta)]R(\theta-\theta'). \tag{3}$$

In addition the Yang–Baxter equation (1) guarantees the factorizability for the S matrix where:

$$S^{ab}_{cd}(\theta)=R^{ab}_{dc}(\theta) \tag{4}$$

§ telnet parthe.lpthe.jussieu

is a two-particle S-matrix in two spacetime dimensions. In this context, $K^-_{ab}(\theta)$ and $K^+_{ab}(\theta)$ describe the scattering of the particles by the left and right boundaries respectively. Therefore for each integrable model (a given solution to the Yang–Baxter equation $R(\theta)$) in order to find the integrable boundary conditions, one must find the solutions $K^-_{ab}(\theta)$ and $K^+_{ab}(\theta)$ of (2) and (3) for the given $R(\theta)$.

We present in this letter the general solution of (2)–(3) for the six vertex model and a family of solutions to the corresponding modifications of these equations for the A_{n-1} vertex model [1]. Also the associated magnetic Hamiltonians are derived and their properties investigated.

Let us first consider the general $K^\pm$ matrices for the six vertex model [1] and its associated Hamiltonians. The R matrix has the form:

$$R(\theta) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \dfrac{\sinh\gamma}{\sinh(\theta+\gamma)} & \dfrac{\sinh\theta}{\sinh(\theta+\gamma)} & 0 \\ 0 & \dfrac{\sinh\theta}{\sinh(\theta+\gamma)} & \dfrac{\sinh\gamma}{\sinh(\theta+\gamma)} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{5}$$

Since this R matrix enjoys P symmetry:

$$PR(\theta)P = R(\theta) \tag{6}$$

equations (2) and (3) are equivalent for the six vertex model. We seek for the general solution of these equations:

$$K(\theta) = \begin{pmatrix} x(\theta) & y(\theta) \\ z(\theta) & t(\theta) \end{pmatrix} \tag{7}$$

where $x(\theta)$, $y(\theta)$, $z(\theta)$ and $t(\theta)$ are unknown functions. Inserting (5) and (7) in (2) or (3) yields ten functional equations for these four functions.

The relevant ones are:

$$z(\theta)y(\theta') = z(\theta')y(\theta) \tag{8}$$

$$\sinh(\theta-\theta')[x(\theta)x(\theta') - t(\theta)t(\theta')] + \sinh(\theta+\theta')[x(\theta')t(\theta) - x(\theta)t(\theta')] = 0 \tag{9}$$

$$y(\theta)x(\theta')\sinh 2\theta' = [\sinh(\theta+\theta')x(\theta) + \sinh(\theta-\theta')t(\theta)]y(\theta'). \tag{10}$$

Equation (8) implies:

$$y(\theta) = k_1 z(\theta) \tag{11}$$

where k_1 is an arbitrary constant.

Equation (9) can be rewritten as:

$$[\tanh(\theta) - \tanh(\theta')][1 - a(\theta)a(\theta')] + [\tanh(\theta) + \tanh(\theta')][a(\theta) - a(\theta')] = 0 \tag{12}$$

where $a(\theta) = t(\theta)/x(\theta)$. Differentiating (12) with respect to θ' and setting $\theta' = 0$ yields an algebraic equation with solution:

$$\frac{t(\theta)}{x(\theta)} = \frac{\sinh(\xi-\theta)}{\sinh(\xi+0)} \tag{13}$$

where ξ is another arbitrary constant. Then, (10) tells us that:

$$y(\theta) = \mu \sinh 2\theta. \tag{14}$$

The remaining equations are identically satisfied. In summary, the general solution $K(\theta)$ for the six-vertex model can be written as:

$$K(\theta, k, \lambda, \mu, \xi) = \begin{pmatrix} k\sinh(\xi+\theta) & \mu \sinh 2\theta \\ \lambda \sinh 2\theta & k \sinh(\xi - \theta) \end{pmatrix} \tag{15}$$

where k, λ, μ and ξ are arbitrary parameters. The special case $\lambda = \mu = 0$ reproduces the solution given in [2].

Let us now discuss the integrable Hamiltonians associated to the K-matrix (15). They follow by the procedure discussed in [2] from the equation:

$$H = C\left\{\sum_{n=1}^{N-1} h_{n,n+1} + \frac{1}{2}\dot{K}_1^-(0) + \frac{\mathrm{tr}_0[K_0^{+t}(-\eta)h_{N0}]}{\mathrm{tr}[K^+(-\eta)]}\right\}. \tag{16}$$

Here C is an arbitrary constant and:

$$h^{n,n+1} = \dot{R}_{n,n+1}(0) \tag{17}$$

gives the two sites Hamiltonian in the bulk and the indices $(n, n+1)$ label the sites in which the matrix acts. In the present case we can choose:

$$K^{\pm}(\theta) = K(\theta, k_{\pm}, \lambda_{\pm}, \mu_{\pm}, \xi_{\pm}). \tag{18}$$

If we want to maintain the bulk part of the XXZ Hamiltonian with the first derivative of the transfer matrix we are led to take K^- matrices with value in $\theta = 0$ different from zero. This leads us to $k_- \neq 0$ and without loss of generality we can take $K^-(0) = 1$, that is $k_- = 1/\sinh\xi$. For the same reason we choose $k_+ \neq 0$.

Inserting (5), (15) and (18) in (16) yields:

$$\begin{aligned} H = \sum_{n=1}^{N-1}&(\sigma_n^x\sigma_{n+1}^x + \sigma_n^y\sigma_{n+1}^y + \cosh\gamma\,\sigma_n^z\sigma_{n+1}^z) \\ &+ \sinh\gamma\,(b_-\sigma_1^z - b_+\sigma_N^z + c_-\sigma_1^- - c_+\sigma_N^- + d_-\sigma_1^+ - d_+\sigma_N^+). \end{aligned} \tag{19}$$

We have chosen $C = 2\sinh\gamma$ and omitted the terms proportional to the identity. The parameters $b_{\pm}$, $c_{\pm}$ and $d_{\pm}$ follow from $\lambda_{\pm}$, $\mu_{\pm}$, $\xi_{\pm}$ and k_+ as shown:

$$\begin{aligned} b_- &= \coth\xi_- \\ b_+ &= \coth\xi_+ \\ c_- &= 2\lambda_- \\ c_+ &= \frac{2\lambda_+}{k_+\sinh\xi_+} \\ d_- &= 2\mu_- \\ d_+ &= \frac{2\mu_+}{k_+\sinh\xi_+}. \end{aligned} \tag{20}$$

The bulk part in (19) is just the well known XXZ Heisenberg Hamiltonian. When $c_\pm = d_\pm = 0$ we recover the Hamiltonian discussed in [2]. Equation (19) provides the more general choice of boundary terms compatible with integrability for the XXZ chain besides periodic and twisted BC. It contains three parameters in each boundary ($b_\pm$, $c_\pm$ and $d_\pm$). In particular when $b_\pm = \pm 1$ and $c_\pm = d_\pm = 0$, one recovers the $SU_q(2)$ invariant Hamiltonian ($q = \mathrm{e}^{\mp\gamma}$) [3, 4].

Let us now consider the $n(2n-1)$ integrable vertex model associated to the A_{n-1} Lie algebra. The R-matrix takes the form (see [1])

$$R_{ij}^{ab}(\theta) = \frac{\sinh\gamma}{\sinh(\gamma+\theta)}\delta_{ia}\delta_{jb}\,\mathrm{e}^{\theta\,\mathrm{sign}(a-b)} + \frac{\sinh\theta}{\sinh(\gamma+\theta)}\delta_{ib}\delta_{ja}$$
$$a \neq b \qquad R_{ii}^{aa} = \delta_{ia} \qquad 1 \leqslant a, b \leqslant n. \tag{21}$$

This reduces to (5) when $n = 2$ upon a gauge transformation [1]. Contrary to the six vertex R-matrix, the R-matrix (21) does not enjoy P and T symmetry but just PT invariance. It is not a crossing invariant either but it obeys the weaker property [5].

$$\left[\left\{[S_{12}(\theta)^{t_2}]^{-1}\right\}^{t_2}\right]^{-1} = L(\theta,\gamma)M_2 S_{12}(\theta+2\eta)M_2^{-1} \tag{22}$$

where $L(\theta,\gamma)$ is a c-number function, η a constant and M a symmetry of the R-matrix (21). That is

$$[M_1 \otimes M_2, R_{12}(\theta)] = 0. \tag{23}$$

We find by direct calculation from (21) and (22):

$$\begin{aligned}
&\eta = \frac{n}{2}\gamma \\
&M_{ab} = \delta_{ab}\,\mathrm{e}^{(n-2a+1)\gamma} \qquad 1 \leqslant a, b \leqslant n \\
&L(\theta,\gamma) = \frac{\sinh(\theta+\gamma)\sinh[\theta+(n-1)\gamma]}{\sinh(\theta)\sinh(\theta+n\gamma)}.
\end{aligned} \tag{24}$$

In this case, when the weak condition (22) holds, the integrability requirements on the boundary matrices $K^-(\theta)$ and $K^+(\theta)$ need some modifications [6]. $K^-(\theta)$ must still obey (2) and $K^+(\theta)$ obeys:

$$R(\theta-\theta')K_1^+(\theta')^{t_1}M_1^{-1}R(-\theta-\theta'-2\eta)K_1^+(\theta)^{t_1}M_2$$
$$= K_1^+(\theta)^{t_1}M_2R(-\theta-\theta'-2\eta)M_1^{-1}K_1^+(\theta')^{t_1}R(-\theta-\theta'). \tag{25}$$

There is an automorphism between K^- and K^+:

$$K^+(\theta) = K^-(-\theta-\eta)^t M. \tag{26}$$

For simplicity, we start searching for diagonal matrices K^-:

$$K_{ab}^-(\theta) = \delta_{ab}K_a^-. \tag{27}$$

Inserting (21) and (27) in (2) yields:

$$\sinh(\theta+\theta')[K_a^-(\theta)K_b^-(\theta')\,\mathrm{e}^{\mathrm{sign}(a-b)(\theta-\theta')} - K_a^-(\theta')K_b^-(\theta)\,\mathrm{e}^{-\mathrm{sign}(a-b)(\theta-\theta')}]$$
$$+\sinh(\theta-\theta')[K_b^-(\theta)K_b^-(\theta)'\,\mathrm{e}^{-\mathrm{sign}(a-b)(\theta+\theta')}$$
$$-K_a^-(\theta')K_a^-(\theta)\mathrm{e}^{\mathrm{sign}(a-b)(\theta'+\theta)}] = 0. \tag{28}$$

These equations are generalizations of (9). By a similar procedure we find their general solution:

$$\begin{aligned} K_a^-(\theta) &= k\sinh(\xi+\theta)\mathrm{e}^{\theta} \qquad && 1 \leqslant a \leqslant l_- \\ K_a^-(\theta) &= k\sinh(\xi-\theta)\mathrm{e}^{-\theta} \qquad && l_-+1 \leqslant a \leqslant n. \end{aligned} \tag{29}$$

Here k, ξ and l_- are arbitrary parameters. For $n = 2$ and $l_- = 1$ we recover the diagonal case of (15) after a gauge transformation. In general, for $n > 2$, we have the extra discrete parameter l_- that tells where the diagonal elements change from one to another.

The integrable Hamiltonian associated to the R-matrix (21) with BC (29) follows using equation (16). We find after some straightforward calculations:

$$H = \sum_{j=1}^{N-1}\left\{\cosh\gamma\sum_{r=1}^{n} e_{rr}^{(j)}e_{rr}^{(j+1)} + \sum_{r,s=1}^{n} e_{rs}^{(j)}e_{sr}^{(j+1)} + \sinh\gamma\sum_{r,s=1}^{n}\mathrm{sign}(r-s)e_{rr}^{(j)}e_{ss}^{(j+1)}\right\}$$
$$+\frac{\sinh\gamma}{2}(1+\coth\xi_-)\left[\sum_{r=1}^{l_-}e_{rr}^{(1)} - \sum_{l_-+1}^{n}e_{rr}^{(1)}\right]$$
$$+\frac{\cosh\gamma}{\mathrm{tr}K^+(0)}\left\{\frac{\sinh(\xi_+-(n\gamma/2))}{\sinh\xi_+}\,\mathrm{e}^{\gamma}\sum_{r=1}^{l_+}e_{rr}^{(N)}\mathrm{e}^{-2r\gamma}\right.$$
$$\left.+\frac{\sinh(\xi_++(n\gamma/2))}{\sinh\xi_+}\,\mathrm{e}^{(n+1)\gamma}\sum_{r=l_+-1}^{n}e_{rr}^{(N)}\mathrm{e}^{-2r\gamma}\right\}$$
$$+\frac{1}{\mathrm{tr}K^+(0)}\sum_{r,s=1}^{n}\mathrm{sign}(r-s)e_{rr}^{(N)}K_s^+(0) \tag{30}$$

where N is the number of sites of the chain and l_+, l_- arbitrary integers running from 1 to n. We have extracted a global factor of $1/\sinh\gamma$ and omitted the terms proportional to the unit operator. In particular with $\xi_\pm = -\infty$ we get a $SU_q(n)$ invariant Hamiltonian, with $q = \mathrm{e}^{-\gamma}$, that can be written as:

$$H = \sum_{j=1}^{N-1}\left\{\sum_{\substack{r,s=1\\ r>s}}^{n}\left(\prod_{l=s}^{r-1}(J_l^+)^{(j)}\prod_{(l=r-1)}^{s}(J_l^-)^{(j+1)} + \prod_{l=r-1}^{s}(J_l^-)^{(j)}\prod_{l=s}^{r-1}(J_l^+)^{(j+1)}\right)\right.$$
$$+\frac{\cosh\gamma}{n}\left[\sum_{\substack{r,s=1\\ r>s}}^{n-1}s(n-r)(h_r^{(j)}h_s^{(j+1)}+h_s^{(j)}h_r^{(j+1)}) + \sum_{r=1}^{n-1}r(n-r)h_r^{(j)}h_r^{(j+1)}\right]$$
$$\left.+\frac{\sinh\gamma}{n}\sum_{\substack{r,s=1\\ r>s}}^{n-1}s(r-s)(n-r)(h_r^{(j)}h_s^{(j+1)}-h_s^{(j)}h_r^{(j+1)})\right\}$$
$$+\frac{\sinh\gamma}{n}\sum_{r=1}^{n-1}r(n-r)(h_r^{(N)}-h_r^{(1)}). \tag{31}$$

Here N is the number of sites, $J_l^+ \equiv e_{ll+1}$, $J_l^- \equiv e_{l+1l}$ and $h_l \equiv e_{ll} - e_{l+1l+1}$ are the $su(n)$ generators in the fundamental representation with $(e_{lm})_{ij} \equiv \delta_{li}\delta_{mj}$. It is easily seen that this Hamiltonian coincides, for $n = 2$, with the $SU_q(2)$ invariant one, discussed in [3, 4].

In conclusion we have obtained the general solution to the surface factorization equations for the six-vertex R matrix providing in this way the more general boundary terms compatible with integrability. The Bethe ansatz in these systems must change drastically as the Hamiltonian does not commute with J_z. For the A_n chain, a generalization of the nested Bethe ansatz [1] will be needed.

AGR would like to thank the LPTHE for the kind hospitality and the Spanish MEC for financial support under grant AP90 02620085. HJ de V would like to thank the Isaac Newton Institute for their kind hospitality.

Note added in proof. After completion of this letter, we heard from A B Zamolodchikov that he has independently obtained equation (15) for the six-vertex model.

References

[1] For a review, see for example, de Vega H J 1983 *Int. Mod. Phys.* A **4** 2371; 1990 *Nucl. Phys.* B *(Proc. Supp.)* 18A 229
[2] Cherednik I 1984 *Theor. Math. Phys.* **61** 35
Sklyanin E K 1988 *J. Phys. A: Math. Gen.* **21** 2375
[3] Pasquieur V and Saleur H 1990 *Nucl. Phys.* B **330** 523
[4] Destri C and de Vega H J 1992 *Nucl. Phys.* B **374** 692; *Nucl. Phys.* B LPTHE preprint 92-08
[5] Reshetikhin N and Semenov-Tian-Shansky M 1990 *L. M. P.* **19** 133
[6] Mezincescu L and Nepomechie R I 1991 *Int. J. Mod. Phys.* A **6** 5231 and addendum